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MATHEMATICAL ANALYSIS OF RATES OF RETURN UNDER CERTAINTY*†

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The purpose of this paper is to prove certain properties of the present and future values of a sequence of cash flows which have applications in the theory of capital budgeting. This is done in Theorems III, IV and V. As an introduction, certain previously available results about the present value function are stated and proved as Theorems I and II. A summary of the relevance of these results in capital budgeting is given in the Summary.

Introduction

We define a firm to be an entity which acquires capital in order to invest it. A project is any possible event or commitment which would change the firm's amount of capital. A project may be completely described by a sequence of $n + 1$ real numbers, assuming that increase or decrease in the amount of capital occurs only at the end of each period during the life of the project, i.e., by a_0, $a_1, \cdots, a_n$. The a_j may be interpreted as the net flow of capital to the firm, i.e., if a_j is negative the flow of capital is *from* the firm, if a_j is positive the flow is *to* the firm. Without loss of generality, it can be assumed that a_0 and a_n are nonzero. Furthermore, it is assumed that not all the a_j's have the same sign.

Let i denote the interest rate, w the compounding rate [$w = 1 + i$] and v the discount rate [$v = (1 + i)^{-1}$]. The discounted cash flow method of evaluating projects is based on the present value function [1]

$$(1) \qquad P(i) = a_0 + a_1v + a_2v^2 + \cdots + a_nv^n$$

or, equivalently, on the future value function

$$(2) \qquad S(i) = a_0w^n + a_1w^{n-1} + \cdots + a_n$$

For most purposes, both functions are equally appropriate since

$$(3) \qquad S(i) = w^nP(i) \quad \text{if} \quad -1 < i < \infty$$

The values of these functions for certain values[4] of i are

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[4] Negative interest rates may be interpreted as "loss of capital." As i decreases to -1, the future value $S(i)$ tends to a_n; all the other cash flows are multiplied by numbers which tend to zero.

i	P	S
-1	$\pm\infty$	a_n
0	$\sum a_j$	$\sum a_j$
∞	a_0	$\pm\infty$

where $\pm\infty$ denotes a functional value which increases without bound.

The *balance* of a project at the end of period j at interest i, is defined as

$$S_j(i) = a_0 w^j + a_1 w^{j-1} + \cdots + a_{j-1} w + a_j$$

The balance may be computed recursively by:

$$(4) \qquad \begin{aligned} S_0(i) &= a_0 \\ S_j(i) &= wS_{j-1}(i) + a_j \qquad j = 1, \cdots, n \end{aligned}$$

We define a project to be a:

pure investment project at interest i if $S_j \leqq 0$ for $j = 0, 1, \cdots n - 1$
pure financing project at interest rate i if $S_j \geqq 0$ for $j = 0, 1, \cdots n - 1$
mixed project at interest rate i if the nonzero S_j for $j = 0, 1, \cdots n - 1$ are not all of one sign.

In practical problems the interest rate for pure investment projects is usually not equal to the interest rate in pure financing projects. Therefore, let

$$r = \text{interest rate in investment projects and } y = 1 + r$$

$$k = \text{interest rate in financing projects and } x = 1 + k$$

Here y and x are the compounding factors corresponding to interest rates r and k, respectively.

The balance of a *mixed* project [2], at the end of period j is a function of x and y defined by the recursive relations:[5]

$$F_0(x, y) = a_0$$

$$(5) \qquad \left.\begin{aligned} F_j(x, y) &= yF_{j-1}(x, y) + a_j \quad \text{if} \quad F_{j-1}(x, y) < 0 \\ &= xF_{j-1}(x, y) + a_j \quad \text{if} \quad F_{j-1}(x, y) > 0 \\ &= a_j \quad \text{if} \quad F_{j-1}(x, y) = 0 \end{aligned}\right\} \qquad j = 1, 2, \cdots, n$$

The function $F(x, y)$ is defined to be $F_n(x, y)$ for $x > 0$, $y > 0$.

Theorems I and II state results for the case $k = r = i$.

Definition I: A *simple* project is one in which the signs of all the nonzero a_j's for $j = 1, 2, \cdots n$ are different from that of a_0.

Theorem IA. The present value function of a simple project in which a_0 is negative is a strictly convex, strictly decreasing function for $i > -1$.

[5] The definition of $F_j(x, y)$ for the case where $F_{j-1}(x, y) = 0$ is arbitrary; it could also be defined by using $F_{j-1}(x, y) \leqq 0$ in the first line of (2) or $F_{j-1}(x, y) \geqq 0$ in the second line.

Proof: $P(i)$ is a polynomial of degree n in v and hence is continuous as a function of i.

From (1)

$$dP(i)/di = -a_1v^2 - 2a_2v^3 - \cdots - na_nv^{n+1}$$

$$<0 \text{ since } a_j \geqq 0 \text{ for } j = 1, 2, \cdots n-1 \text{ and } a_n > 0, \text{ by assumption, and } v > 0$$

$$d^2P(i)/di^2 = 2a_1v^3 + (3)(2)a_2v^4 + \cdots + (n+1)(n)a_nv^{n+2}$$

$$>0$$

Since the first derivative is negative the function is strictly decreasing and since the second derivative is positive the function is strictly convex.

Corollary IB: The present value function for a simple project in which a_0 is positive is a strictly concave, strictly increasing function for $i > -1$.

Corollary IC: The equation $P(i) = 0$ for a simple project has a single unique solution for $i > -1$. The solution is negative if $\sum a_j$ has the same sign as a_0, positive if $\sum a_j$ has a sign different from a_0, and zero if $\sum a_j$ is zero ($j = 0, 1, 2, \cdots n$).

Proof: If a_0 is negative, a_n must be positive since the project is simple. The

$$\lim_{i\to -1} P(i)$$

is dominated by the term a_nv^n, hence is positive. Since $P(i)$ is continuous and is positive for i close to -1 and negative for large i, there must be a root >-1. However, $P(0) = \sum a_j$ and $\lim_{i\to\infty} P(i) = a_0 < 0$. Hence if $\sum a_j$ is positive the root is greater than zero, and if $\sum a_j$ is negative the root is less than zero. The proof for positive a_0 is analogous.

Theorem IIA.[6] The present value function of a project which is a pure investment for all $i > -1$ is a strictly decreasing function for $i > -1$.

Proof: The function is a polynomial in v, hence is a continuous function of i. The proof is by induction. From (4)

$$dS_0(i)/di = 0; \qquad dS_1/di = (d/di)\{(1+i)S_0(i) + a_1\}$$
$$= S_0(i) + (1+i)dS_0(i)/di$$

which is negative, since, by assumption, $S_0(i)$ is negative and $dS_0(i)/di$ is zero.

Now suppose that $dS_j/di < 0$, then

$$dS_{j+1}/di = S_j(i) + (1+i)dS_j(i)/di < 0$$

since the first term is either zero or negative and the second term is negative.

Consequently, by induction, $dS_n(i)/di < 0$ and $S_n(i)$ is strictly monotonically decreasing. By (3) $P(i)$ is strictly monotonically decreasing.

[6] The conditions of Theorem IIA are met only when $a_0, a_1, \cdots a_{n-1}$ are all $\leqq 0$. The general case is that of Corollary IID.

Corollary IIB. The present value function of a pure financing project is a strictly increasing function for $i > -1$.

Corollary IIC, [3]. The equation $P(i) = 0$ has a single unique solution for $i > -1$ for pure investment and pure financing projects. The solution is negative if $\sum a_j$ has the same sign as a_0, positive if $\sum a_j$ has a different sign from a_0 and zero if $\sum a_j$ is zero.

Proof: The proof is analogous to that of Corollary IC.

Corollary IID. If a project is a pure investment project for some interest rate $i_{\min} > -1$ then $P(i)$ is a strictly decreasing function of i for $i > i_{\min}$.

Proof: The proof is the same as for Theorem IIA.

Corollary IIE.

Any project in which a_0 is negative is a pure investment project for some $i_{\min}$.

Theorems III and IV state results for a present value as a function of two interest rates x and y (or k and r).

Theorem III. The future value function $F(x, y)$ of a mixed project, defined by (5), has the following properties for $0 < x < \infty$; $0 < y < \infty$.

1. $F(x, y)$ is a polynomial in x, y. The degree depends on (x, y).
2. (a) If $a_0 < 0$, there exists a $y_{\min}$ such that $F(x, y)$ for $y > y_{\min}$ is of zero degree in x.
   The region $0 < y < y_{\min}$, $x > 0$ is termed the *mixed* region
   The region $y_{\min} \leqq y < \infty$, $x > 0$ is termed the *pure investment* region
   (b) If $a_0 > 0$, there exists an $x_{\min}$ such that $F(x, y)$ for $x > x_{\min}$ is of zero degree in y.
   The region $y > 0$, $0 < x < x_{\min}$ is termed the *mixed* region
   The region $y > 0$, $x_{\min} \leqq x < \infty$ is termed the *pure financing* region
3. $F(x, y)$ is a continuous function of (x, y).
4. The partial derivatives $\partial F/\partial x$, $\partial F/\partial y$, exist at all points in the mixed region except for the points at which one or more project balances are zero.

Proof:

1. The function $F(x, y)$

Carrying out the recursions (5) for a given project produces a polynomial in x and y of the form

$$F(x, y) = F_n(x, y) = a_0 x^{\alpha_0} y^{\beta_0} + a_1 x^{\alpha_1} y^{\beta_1} + \cdots a_{n-1} x^{\alpha_{n-1}} y^{\beta_{n-1}} + a_n$$

The degrees of the polynomial in x and y at the point (x, y) are the exponents α_0, β_0. In the sequence of values $F_0, F_1, \cdots F_{n-1}$ calculated by the recursive process (5), α_0 is the number of positive values and β_0 is the number of negative values. Similarly, α_1 is the number of positive values, and β_1 the number of negative values, in the sequence $F_1, F_2, \cdots F_{n-1}$; α_2 and β_2 are numbers of the same kind for the sequence $F_2, F_3, \cdots F_{n-1}$, and so on. If F_{n-1} is positive, $\alpha_{n-1} = 1$ and $\beta_{n-1} = 0$; if it is negative, $\alpha_{n-1} = 0$ and $\beta_{n-1} = 1$.

Thus, at any point (x, y), $F(x, y)$ is represented by a polynomial in x and y whose coefficients are the numbers $a_0 \cdots a_n$ and whose exponents are determined by the signs of the project balances $F_0, F_1, \cdots F_{n-1}$. At any point (x, y)

where none of the project balances is zero, the following equations hold:

$$\alpha_j + \beta_j = n - j \qquad j = 0, 1, \cdots n - 1$$

At a point (x, y), where one or more of the project balances is zero, $a_0 + \beta_0$ will be less than n by the number of zero project balances, and the other sums, $\alpha_i + \beta_i$ will similarly be decreased by the number of zeros in the corresponding subsequences $F_i, F_{i+1}, \cdots F_{n-1}$.

From the continuity of polynomials, it follows that any point at which

$$\alpha_0 + \beta_0 = n$$

is surrounded by a neighborhood of points at which $\alpha_0 + \beta_0 = n$ and we can asssert that

Any point (x_0, y_0), at which none of the $F_0, F_1, \cdots F_{n-1}$ is zero, is surrounded by a neighborhood of points throughout which $F(x, y)$ is represented by the same polynomial as it is at (x_0, y_0).

2. Pure and Mixed Regions.

If a_0 is positive, it is evident that, for large enough values of x, all of the project balances will be positive. There exists, therefore, a smallest value of x, say $x_{\min}$, which makes at least one of the partial evaluations zero and all the others positive. For values of x greater than or equal to $x_{\min}$, $F(x, y)$ is a function of x only. Typical contour lines of the function $F(x, y)$ for $a_0 > 0$ are shown in Figure 1(a).

For $x \geqq x_{\min}$ the contour lines of the function $F(x, y)$ are parallel to the y axis.

Similarly, *if a_0 is negative,* there exists a smallest value of y, say $y_{\min}$, such that, for values of y greater than or equal to $y_{\min}$, $F(x, y)$ is a function of y

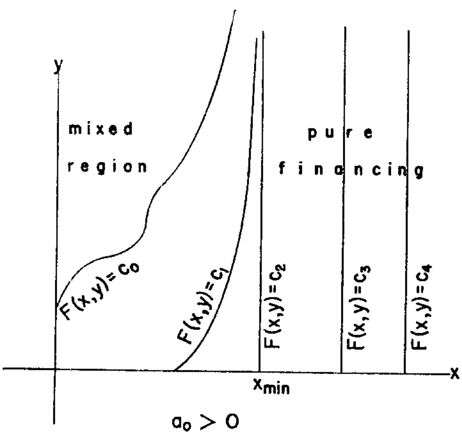

FIGURE 1(a)
CONTOUR LINES OF F(x,y)=c
$c_0 < c_1 < c_2 < c_3 < c_4$

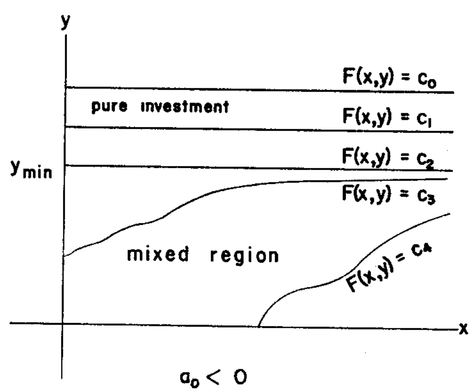

FIGURE 1(b)
CONTOUR LINES OF F(x,y)=c
$c_0 < c_1 < c_2 < c_3 < c_4$

only. Typical contour lines are shown in Figure 1(b). For $y \geqq y_{\min}$ the contour lines are straight lines parallel to the x axis.

The region in the positive x, y quadrant in which $F(x, y)$ is a function of x and y is defined to be the mixed region.

3. Continuity of $F(x, y)$

The continuity of $F(x, y)$, computed by the recursion relation (5), can be established by induction. $F_0(x, y)$ is a continuous (in fact, constant) function of x and y. If $F_{j-1}(x, y)$ is a continuous function of x and y then clearly $F_j(x, y)$ is also a continuous function of x and y at all except possibly at points at which $F_{j-1}(x, y) = 0$. However, the limit at any such point through any sequence of points is a_j, hence the function is also continuous at these points. Therefore F_j is continuous for all points if F_{j-1} is, and by induction $F_n(x, y) = F(x, y)$ is continuous.

4. Partial Derivatives

By assumption there is at least one $F_j(x, y)$ which is positive for (x, y) in the mixed region; let F_m be the first one. Then $F_0, F_1, \cdots F_{m-1}$ are negative and do not depend on x. Therefore

$$\partial F_0/\partial x = \partial F_1/\partial x = \cdots = \partial F_m/\partial x = 0$$

However

$$\begin{aligned}\partial F_{m+1}/\partial x &= (\partial/\partial x)\{xF_m(x, y) + a_j\}\\ &= F_m(x, y) + x(\partial F_m(x, y)/\partial x)\end{aligned}$$

Hence

$$\partial F_{m+1}/\partial x$$

is positive since it is the sum of a positive and zero term.

Now suppose

$$\partial F_j/\partial x > 0 \quad \text{and consider} \quad \partial F_{j+1}/\partial_x . \qquad j \geqq m + 1.$$

From (5)

$$\begin{aligned}\partial F_{j+1}/\partial x &= y(\partial F_j/\partial x)(>0 \text{ because } \partial F_j/\partial x > 0 \text{ by assumption}) && \text{if} \quad F_j < 0\\ &= F_j + x(\partial F_j/\partial x)(>0 \text{ since both terms are } >0) && \text{if} \quad F_j > 0\end{aligned}$$

$$\partial F_{j+1}/\partial x \text{ does not exist} \qquad \text{if} \quad F_j = 0$$

The partial derivative $\partial F_{j+1}/\partial x$ is positive if it exists and consequently, if the succeeding partial derivatives exist, $\partial F_n/\partial x = \partial F(x, y)/\partial x$ is positive.

An exactly analogous proof shows that if all the partial derivatives $\partial F_j/\partial y$ exist, then $\partial F_n/\partial x = \partial F(x, y)/\partial y$ exists and is negative. Neither partial derivative $\partial F/\partial x$, $\partial F/\partial y$ will exist at a point x, y at which one of the F_j is zero.

At the points where the partial derivatives do not exist, the right-hand and left-hand derivatives are not equal, because the degree of the polynomial changes at points where an $F_j = 0$. However, both derivatives are of the same sign and any line $x = c$ or $y = c$, i.e., any line parallel to either axis, contains only a finite number of such points.

Theorem IV. The function $y = y(x)$, implicitly defined in the mixed region by

$$F(x, y) = 0$$

is continuous and strictly increasing for

$$x_L < x < x_R$$

where x_L and x_R depend on a_0 and a_n as follows:

	$a_0 < 0$	$a_0 > 0$	
		$F(x_{\min}, y) > 0$	$F(x_{\min}, y) < 0$
$a_n < 0$	$x_L > 0;\ x_R = \infty$	$x_L > 0;\ x_R = x_{\min}$	y is not a function of x
$a_n > 0$	$x_L = 0;\ x_R = \infty$	$x_L = 0;\ x_R = x_{\min}$	not possible

Proof: The continuity of $y(x)$ follows directly from the continuity of $F(x, y)$ given above. Furthermore, if a function $F(x, y)$ is continuous and possesses partial derivatives in a region, the derivative dy/dx of the function $y(x)$ defined implicitly by $F(x, y) = 0$ is given by

$$\frac{dy}{dx} = -\frac{\partial F}{\partial x} \Big/ \frac{\partial F}{\partial y}$$

From Theorem III it follows that dy/dx exists and is positive for all x, y in the mixed region except at points for which one or more of the project balances are zero. Hence the function $y(x)$ is strictly increasing in some interval.

The x-interval over which $y(x)$ is defined depends on a_0 and a_n, because $F(0, 0)$, the value at the origin, is a_n and the sign of a_0 determines whether $F(x, y)$ has a pure financing or a pure investing region.

If a_0 is negative then the project becomes a pure investment project for $y > y_{\min}$ (Fig. 1b). Hence x_L will be zero if a_n is positive and will be greater than zero if a_n is negative since $F(x, 0)$ is strictly increasing in x and x_L is the root of $F(x, 0) = 0$. Since, in a mixed region, $dy/dx > 0$, $y(x)$ is asymptotic to the horizontal straight line $y = F(x, y_{\min})$ as x becomes large.

If a_0 is positive, then the project becomes a pure financing project for $x > x_{\min}$ (Fig. 1a). If $F(x_{\min}, y)$ is zero or negative, $F(x, y) = 0$ is a straight line parallel to the y-axis and y is not defined as a function of x. If $F(x_{\min}, y)$ is positive and a_n is negative, then x_L is between zero and $x_{\min}$. If a_n is positive, x_L is zero. The function $y(x)$ is asymptotic to the vertical straight line $x = F(x_{\min}, y)$ as x approaches $x_{\min}$.

Corollary IVA. Similar properties hold for the function $y(x;\, c)$ defined implicitly by $F(x, y) = c$.

Proof: $F(x, y) = c$ may be written in the form

$$F(x, y) - c = 0$$

The function to the left of the equal sign differs from $F(x, y)$ only in the coefficient a_n. The functions $y(x;\, c)$ are asymptotic to $y = F(x, y_{\min})$ if $a_0 < 0$ and $F(x, y_{\min}) < c$ and are asymptotic to $x = F(x_{\min}, y)$ if $a_0 > 0$ and $F(x_{\min}, y) > c$.

Corollary IVB. Results analogous to those in Theorem IV and Corollary IVA hold for $x = x(y)$ defined implicitly by

$$F(x, y) = 0$$

Theorem V. For any project $P(i)$ defined by (1) and $y(x)$ defined by Theorem IV:

$$P(i) > 0 \quad \text{if} \quad y(x) > x$$

$$P(i) = 0 \quad \text{if} \quad y(x) = x$$

$$P(i) < 0 \quad \text{if} \quad y(x) < x$$

Proof: Let

$$x = 1 + i$$

Then the Present Value $P(i)$ is given as a function of x by

$$P(x) = a_0 + a_1/x + a_2/x^2 + \cdots a_n/x^n$$

Multiplying both sides by x^n gives

$$x^n P(x) = a_0 x^n + a_1 x^{n-1} + \cdots a_n = F(x, x)$$

$F(x, x)$ is merely the function $F(x, y)$ evaluated for points along the line $y = x$. For positive x, it will obviously be negative where $P(x)$ is negative, zero where $P(x)$ is zero, and positive where $P(x)$ is positive.

Summary

There are two widely discussed methods for making decisions about the acceptance or rejection of projects on the basis of the discounted cash flows. The internal rate of return method accepts a project only if the solution of $P(i) = 0$ (equation 1) is greater than some stated rate i_0. The present value method accepts the project only if $P(i_0)$ is positive. The internal rate of return method cannot be used if the equation $P(i) = 0$ has more than one solution. The present value method can always be applied but in some projects $P(i)$ increases as i increases. This is contrary to intuition and the reason why it occurs is not readily apparent.

From the analysis in this paper it is now clear that the difficulties with both methods arise in the mixed region. If at compounding rates (x, y) a project falls in the mixed region, then forcing it into the form of $P(i)$ in (1) is equivalent to considering only points on the line $x = y = 1 + i$. In the mixed region a project is sometimes an investment and sometimes a source of funds, and hence the analysis based on $P(i)$ is equivalent to assuming that the *rate the firm imputes to the project when it is a source of funds is the same rate as the rate the project earns when there is an investment.* Theorem IV states that for any project, the rate of return which a project can earn $(y - 1)$ will increase as the rate $(x - 1)$ imputed or credited to the project increases. Theorem V verifies that $P(i)$ is in fact equivalent to $F(x, y)$ for the special case where $x = y = 1 + i$.

A detailed examination of the implications of these results for capital budgeting theory is given in [4]. It is shown that increasing the present value of the firm requires the use of rules which may be stated as:

$$\text{accept project if} \quad y(u) > u \quad \text{or} \quad u > x(u)$$

where $y(x)$ and $x(y)$ are the functions implicitly defined by $F(x, y) = 0$ and $u = 1 + i_0$. Since i_0 (often referred to as the cost of capital) is assumed given, mixed projects may be considered either as financing projects [$x(u)$] or as investment projects [$y(u)$]. In either case the decision is the same. This rule is exactly equivalent to the discounted present value rule if the traditional present value function $P(i)$ is used.

Acknowledgement

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